

Progress in relativistic gravitational theory using the inverse scattering method

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Abstract

The increasing interest in compact astrophysical objects (neutron stars, binaries, galactic black holes) has stimulated the search for rigorous methods, which allow a systematic general relativistic description of such objects. This paper is meant to demonstrate the use of the inverse scattering method, which allows, in particular cases, the treatment of rotating body problems. The idea is to replace the investigation of the matter region of a rotating body by the formulation of boundary values along the surface of the body. In this way we construct solutions describing rotating black holes and disks of dust (“galaxies”). Physical properties of the solutions and consequences of the approach are discussed. Among other things, the balance problem for two black holes can be tackled.

I. INTRODUCTION

The systematic investigation of neutron stars and binaries consisting of pulsars and other compact objects and the increasing evidence for the existence of (rotating) black holes have stimulated theoretical and numerical studies on rapidly rotating bodies in General Relativity. No doubt, realistic stellar models (e.g. neutron star models) require a careful physical analysis of their interior states and processes and, as a consequence, extensive *numerical* calculations. On the other hand, there is widespread interest for explicit solutions of the rotating body problem under simplifying assumptions. Such solutions could provide a deeper insight into physical phenomena connected with spinning matter configurations and, moreover, serve as test beds for the numerical investigations mentioned before. A good example is the Kerr solution, which has enriched our knowledge of rotating black holes in an inestimable way. However, rigorous results for rotating bodies are relatively rare in General Relativity. Among other things, this is due to the mathematical difficulties with ‘free boundary value problems’, already known from Newton’s gravitational theory, and to the specific complexity of the differential equations of Einstein’s theory inside the body. Namely, the shape of the surface of a rotating self-gravitating fluid ball — the best model for astrophysical applications — is a ‘compromise’ between gravitational and centrifugal forces and not known *a priori*. (The surface is ‘free’, i.e. not fixed from the very beginning.) Though there are powerful (soliton-) techniques to generate (formal) stationary axisymmetric solutions of Einstein’s vacuum equations, no algorithm to integrate the interior field equations is available. Hence, at first glance, a boundary value description of rotating bodies seems to be questionable and inadequate. However, there are exceptional cases, in which the surface of the body has a known shape and the surface values provide enough information to construct the complete solution of the vacuum field equations. It is the intention of this paper to show that this is true for stationary black holes and disks of dust, which may be considered to be extremely flattened perfect fluid bodies. Moreover, it should become clear that our procedure, which is based on the inverse scattering method, opens an access to the not yet solved problem of the balance of two black holes and enables, in principle, the construction of black holes surrounded by dust rings (‘AGN models’).

Another interesting domain of application for the inverse scattering method is colliding gravitational waves. This theory is out of the scope of our paper. We refer to the article¹

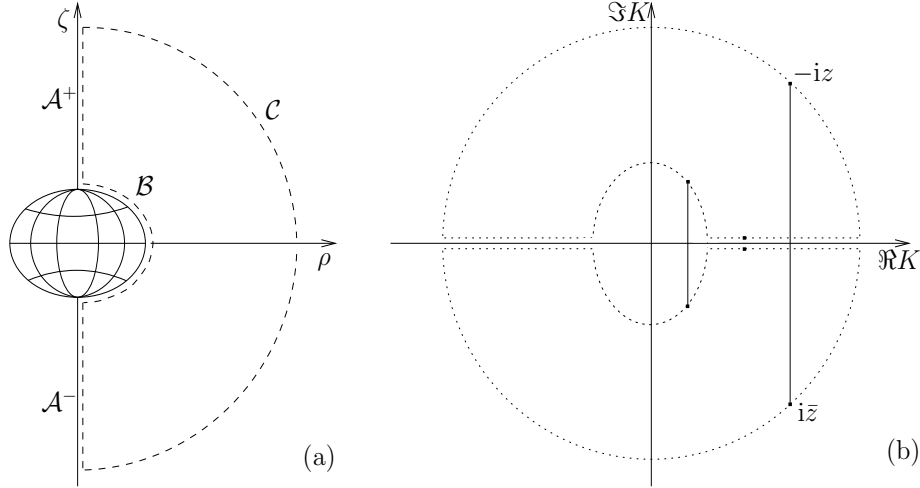


FIG. 1: (a) A slice $\phi = \text{constant}, t = \text{constant}$, (b) 2-sheeted Riemann K -surface with branch points (dots) and two cuts (solid lines)²

and references therein.

The present work is mainly based on the papers² and³ but it also contains substantial material not published before.

II. THE BOUNDARY VALUE PROBLEM

We consider a simply connected axisymmetric and stationary body and describe its exterior vacuum gravitational field in Weyl-Lewis-Papapetrou coordinates

$$ds^2 = e^{-2U} [e^{2k} (d\rho^2 + d\zeta^2) + \rho^2 d\phi^2] - e^{2U} (dt + a d\phi)^2 \quad (1)$$

where the ‘Newtonian’ gravitational potential U and the ‘gravitomagnetic’ potential a are functions of ρ and ζ alone. Fig. 1(a) shows the boundaries of the vacuum region: \mathcal{A}^\pm are the regular parts of the axis of symmetry ($\rho = 0$), \mathcal{B} is the surface of the body and \mathcal{C} stands for spatial infinity. Later on, we will integrate along the dashed line and pick up information from the boundary values of the gravitational fields at \mathcal{A}^\pm , \mathcal{B} and \mathcal{C} . The metric (1) allows an Abelian group of motions G_2 with the generators (Killing vectors)

$$\begin{aligned} \xi^i &= \delta_t^i, & \xi^i \xi_i &< 0 & \quad (\text{stationarity}) \\ \eta^i &= \delta_\phi^i, & \eta^i \eta_i &> 0 & \quad (\text{axisymmetry}) \end{aligned} \quad (2)$$

where the Kronecker symbols δ_t^i and δ_ϕ^i indicate that ξ^i has only a t -component whereas η^i points in the azimuthal ϕ -direction (its trajectories are closed circles!). Obviously,

$$e^{2U} = -\xi^i \xi_i, \quad a = -e^{-2U} \eta_i \xi^i \quad (3)$$

is a coordinate-free representation of the two relativistic gravitational fields U (generalization of the Newtonian gravitational potential) and a (gravitomagnetic potential). To get a unique definition of U and a , we prescribe their behaviour at infinity. Assuming that the space-time has to be flat at large distances from the body and can be described by a Minkowskian line element (1) in cylindrical coordinates, we are led to the boundary conditions

$$\mathcal{C} : \quad U \rightarrow 0, \quad a \rightarrow 0, \quad k \rightarrow 0 \quad (4)$$

Any linear transformation

$$t' = t, \quad \phi' = \phi - \omega t \quad (5)$$

introduces a frame of reference which rotates with a constant angular velocity ω with respect to that asymptotic Minkowski space.

To describe stationarity and axisymmetry in that rotating system one would use the Killing vectors

$$\tilde{\xi}^i = \xi^{i'} + \omega \eta^{i'}, \quad \tilde{\eta}^i = \eta^{i'}, \quad (6)$$

instead of (2).

Regularity of the metric along \mathcal{A}^\pm means

$$\mathcal{A}^\pm : \quad a = 0, \quad k = 0. \quad (7)$$

These conditions express the fact that \mathcal{A}^\pm is an axis of symmetry ($a = 0$) and ensure elementary flatness along the axis ($k = 0$). The behaviour of U and a at the surface \mathcal{B} of the body depends on the physical nature of it. Rotating *perfect fluids* are characterized by a *four-velocity field* u^i consisting of a linear combination of the two Killing vectors,

$$u^i = e^{-V} (\xi^i + \Omega \eta^i) \quad , \quad u^i u_i = -1 \quad (8)$$

where Ω is the angular velocity of the body, and an *invariant scalar pressure* p , which is, for rigid rotation,

$$\Omega = \Omega_0, \quad (\Omega_0 \text{ a constant}) \quad (9)$$

a function of V alone,

$$p = p(V), \quad (10)$$

as a consequence of the Euler equations. Along the surface of the body (if it exists) the pressure has to vanish,

$p(V_0) = 0$, i. e. V must be a constant along \mathcal{B} ,

$$\mathcal{B}: \quad e^{2V} \equiv -(\xi^i + \Omega_0 \eta^i)(\xi_i + \Omega_0 \eta_i) = e^{2V_0} \quad (11)$$

That is a further boundary condition. When identifying ω in (5) and Ω_0 we introduce a frame of reference co-rotating with the body and may interpret V as the co-rotating ‘Newtonian’ potential, cf. (3) and (11). Interestingly, the event horizon \mathcal{H} of a stationary (axisymmetric) black hole behaves like an ‘ordinary’ perfect fluid surface (11). Namely, one can show that a linear combination of the two Killing vectors, $\xi^i + \Omega_H \eta^i$ has a vanishing norm along \mathcal{H} ,

$$\mathcal{H}: \quad e^{2V} \equiv (\xi^i + \Omega_H \eta^i)(\xi_i + \Omega_H \eta_i) = 0, \quad (12)$$

where Ω_H is the angular velocity of the horizon. Hence we may include black holes in our scheme, see Fig. 1(a), for $V_0 \rightarrow -\infty$ and $\mathcal{H} \equiv \mathcal{B}$. It will turn out that (12) together with the correct positioning of the horizon \mathcal{H} in Weyl-Lewis-Papapetrou coordinates together with the asymptotic behaviour (4) of the (invariant) potentials (3) suffices for an explicit construction of the Kerr solution - thus providing a simple constructive uniqueness proof for stationary axisymmetric black holes. On the other hand, the condition (11) is not sufficient to calculate the gravitational vacuum field of rotating *perfect fluid balls*. However, in the disk of dust limit of such fluid configurations the field equations themselves will provide the missing boundary condition along the surface \mathcal{B} of the disk, see³. Starting with that completed set of boundary conditions we will be able to construct the global solution for the rigidly rotating disk of dust.

III. THE FIELD EQUATIONS

The vacuum Einstein equations for the metric coefficients k , U , a are equivalent to the Ernst equation

$$(\Re f) \left(f_{,\rho\rho} + f_{,\zeta\zeta} + \frac{1}{\rho} f_{,\rho} \right) = f_{,\rho}^2 + f_{,\zeta}^2 \quad (13)$$

for the complex function

$$f(\rho, \zeta) = e^{2U} + ib, \quad (14)$$

where b replaces a via

$$a_{,\rho} = \rho e^{-4U} b_{,\zeta}, \quad a_{,\zeta} = -\rho e^{-4U} b_{,\rho} \quad (15)$$

and k can be calculated from

$$k_{,\rho} = \rho \left[U_{,\rho}^2 - U_{,\zeta}^2 + \frac{1}{4} e^{-4U} (b_{,\rho}^2 - b_{,\zeta}^2) \right], \quad k_{,\zeta} = 2\rho \left[U_{,\rho} U_{,\zeta} + \frac{1}{4} e^{-4U} b_{,\rho} b_{,\zeta} \right]. \quad (16)$$

As a consequence of the Ernst equation (13), the integrability conditions $a_{,\rho\zeta} = a_{,\zeta\rho}$ and $k_{,\rho\zeta} = k_{,\zeta\rho}$ are automatically satisfied such that the metric functions a and k may be calculated via line integration from the Ernst potential f . Thus, it is sufficient to discuss the Ernst equation alone.

IV. THE LINEAR PROBLEM

The existence of a Linear Problem (LP) for the Ernst equation^{4,5,6,7,8,9} is the cornerstone of our analysis since it provides a suitable instrument for tackling boundary value problems: the inverse scattering method (ISM). Here we will use a ‘local’ version¹⁰ of the Linear Problem,

$$\begin{aligned} \Phi_{,z} &= \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} + \lambda \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\} \Phi, \\ \Phi_{,\bar{z}} &= \left\{ \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{B} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{A} \\ \bar{B} & 0 \end{pmatrix} \right\} \Phi, \end{aligned} \quad (17)$$

where $\Phi(z, \bar{z}, \lambda)$ is a 2×2 matrix depending on the spectral parameter

$$\lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}} \quad (18)$$

as well as on the complex coordinates $z = \rho + i\zeta$, $\bar{z} = \rho - i\zeta$, whereas A, B and the complex conjugate quantities \bar{A}, \bar{B} are functions of z, \bar{z} (or ρ, ζ) and do not depend on K . From the integrability condition and the formulae

$$\lambda_{,z} = \frac{\lambda}{4\rho} (\lambda^2 - 1), \quad \lambda_{,\bar{z}} = \frac{1}{4\rho\lambda} (\lambda^2 - 1) \quad (19)$$

it follows that a matrix polynomial in λ has to vanish. This yields the set of first order differential equations

$$A_{,\bar{z}} = A(\bar{B} - \bar{A}) - \frac{1}{4\rho}(A + \bar{B}), \quad B_{,\bar{z}} = B(\bar{A} - \bar{B}) - \frac{1}{4\rho}(B + \bar{A}). \quad (20)$$

The system has the ‘first integrals’

$$A = \frac{f_{,z}}{f + \bar{f}}, \quad B = \frac{\bar{f}_{,z}}{f + \bar{f}}. \quad (21)$$

Resubstituting A and B in the equations (20) one obtains the Ernst equation (13). Thus, the Ernst equation is the integrability condition of the LP (17). Vice versa, if f is a solution to the Ernst equation, the matrix Φ calculated from (17) does not depend on the path of integration. The idea of the inverse scattering method (ISM) is to discuss Φ , for fixed but arbitrary values of z, \bar{z} , as a holomorphic function of λ (or K) and to calculate A, B and finally f afterwards. To obtain the desired information about the holomorphic structure in λ , we will integrate the Linear System along the dashed line in Fig. 1(b) making use of the conditions (4), (7), (11) or (12). In this way, we will solve the *direct problem* of the ISM and obtain $\Phi(z, \bar{z}, \lambda)$ for $z, \bar{z} \in \mathcal{A}^\pm, \mathcal{B}, \mathcal{C}$. It turns out that the holomorphic structure remains unchanged by an extension of z, \bar{z} off the axis of symmetry into the entire vacuum region such that one can construct functions Φ with prescribed properties in λ from which one obtains the desired solution $f(z, \bar{z})$ everywhere in the vacuum region. This second step can be very technical and will, in general, lead to linear integral equations for Φ . In some circumstances, λ may be replaced by K . For this purpose, it may be helpful to discuss the mapping (18) of the two-sheeted Riemann surface of K onto the λ -plane for different values of ρ, ζ (or equivalently z, \bar{z}). Fig. 1(b) shows the position of the branch points $K_B = i\bar{z}, \bar{K}_B = -iz$ for the marked path $\mathcal{A}^+ \mathcal{C} \mathcal{A}^- \mathcal{B}$ of Fig. 1(a). It reflects the slice $\phi = \text{constant}, t = \text{constant}$ (Fig. 1(a)) and indicates, in particular, the position and shape of the body. Note that Φ is not defined in the non-vacuum domain inside the circular contour around the origin.

Consider now a Riemann surface with confluent branch points $K_B = \bar{K}_B = \zeta \in \mathcal{A}^+$. Here λ degenerates and takes the values $\lambda = -1$ for K ’s in the lower sheet, say, and $\lambda = +1$ for K ’s in the upper sheet ($K \neq K_B$).

We will now travel along the dashed line of Fig. 1(a) starting from and returning to any point $\rho = 0, \zeta \in \mathcal{A}^+$. (In Fig. 1(b) this corresponds to the bold faced points on the real axis.) Note that $\lambda = -1$ for all K ’s ($K \neq \zeta$) in the lower and $\lambda = +1$ for all K ’s ($K \neq \zeta$)

in the upper sheet of the Riemann K -surface belonging to axis values $\rho = 0$, $\zeta \in \mathcal{A}^\pm$ (the corresponding branch points cling to either side of the real axis in Fig. 1(b)). For $\rho, \zeta \in \mathcal{C}$, the cut between the branch points (e.g., right solid line in Fig. 1(b)) points over the entire K -surface and puts ‘upper’ K values into the lower sheet and ‘lower’ K values into the ‘upper’ sheet. As a consequence, λ will change from ± 1 to ∓ 1 between $\rho = 0$, $\zeta = +\infty$ and $\rho = 0$, $\zeta = -\infty$ ¹¹. This ‘exchange of sheets’ is important for the solution of the linear problem: The initial value $\Phi(\rho_0, \zeta_0, \lambda)$ can (and must) be fixed only in *one* sheet of the K -surface. The dependence on K in the other sheet follows by integration of the LP (17) along a suitable path¹¹.

We will divide the integration of the LP (17) along the closed dashed line of Fig. 1(a) into two steps:

- (i) Integrating along $\mathcal{A}^+\mathcal{C}\mathcal{A}^-$

This step can be performed without particular knowledge about the body and leads to a “*general* solution” for Φ on the regular parts \mathcal{A}^\pm of the symmetry axis.

- (ii) Integrating along \mathcal{B}

Here we confine ourselves to black holes and disks of dust.

V. SOLUTION OF THE DIRECT PROBLEM

A. Axis and Infinity

Without loss of generality the matrix Φ may be assumed to have the structure

$$\Phi = \begin{pmatrix} \psi(\rho, \zeta, \lambda) & \psi(\rho, \zeta, -\lambda) \\ \chi(\rho, \zeta, \lambda) & -\chi(\rho, \zeta, -\lambda) \end{pmatrix} \quad (22)$$

together with

$$\overline{\psi\left(\rho, \zeta, \frac{1}{\lambda}\right)} = \chi(\rho, \zeta, \lambda). \quad (23)$$

Note that both columns of Φ are independent solutions of (17). The particular form of (22) is equivalent to

$$\Phi(-\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi(\lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (24)$$

For $K \rightarrow \infty$ and $\lambda = -1$ the functions ψ, χ may be normalized by

$$\psi(\rho, \zeta, -1) = \chi(\rho, \zeta, -1) = 1 \quad (25)$$

Finally, the solution to the Ernst equation can be read off at $\lambda = 1$ ($K \rightarrow \infty$),

$$f(\rho, \zeta) = \chi(\rho, \zeta, 1), \quad \left(\overline{f(\rho, \zeta)} = \psi(\rho, \zeta, 1) \right). \quad (26)$$

Remarkably enough, the Ernst equation retains its form in the frame of reference co-rotating with the body ($\omega = \Omega_0$). This is a consequence of (3) and (6) and implies the existence of a Linear Problem (17) in the co-rotating system. In particular, the Φ -matrices of both systems of reference are connected by the relation

$$\begin{aligned} \Phi' = & \left[\begin{pmatrix} 1 + \Omega_0 a - \Omega_0 \rho e^{-2U} & 0 \\ 0 & 1 + \Omega_0 a + \Omega_0 \rho e^{-2U} \end{pmatrix} \right. \\ & \left. + i(K + iz)\Omega_0 e^{-2U} \begin{pmatrix} -1 & -\lambda \\ \lambda & 1 \end{pmatrix} \right] \Phi. \end{aligned} \quad (27)$$

Henceforth, a prime marks ‘co-rotating’ quantities. We can now realise our programme and integrate the Linear Problem (17) along the part $\mathcal{A}^+ \mathcal{C} \mathcal{A}^-$ of the dashed line in Fig. 1(a). Using (17) along \mathcal{A}^\pm and (21) one finds for the axis values of Φ

$$\mathcal{A}^+ : \quad \Phi = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix} \quad (28)$$

$$\mathcal{A}^- : \quad \Phi = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & G(K) \\ 0 & F(K) \end{pmatrix}, \quad (29)$$

where $f(\zeta) = f(\rho = 0, \zeta)$ is the axis value of the Ernst potential and $F(K), G(K)$ are integration ‘constants’ depending on K alone. The particular form of (28) is due to the initial condition $\psi = \chi = 1$ for some $\rho_0 = 0, \zeta = \zeta_0 \in \mathcal{A}^+$, $\lambda = -1$ (K in the lower sheet), which fixes the second column of Φ in (28), cf. (22). The first column corresponds to the upper ($\lambda = 1$) sheet and represents a general integral with the two integration ‘constants’ $F(K), G(K)$ which cannot be specified here. Along \mathcal{C} , $\Phi = \Phi(K)$ does not depend on ρ and ζ , since A and B vanish, cf. (21). The ‘exchange of sheets’ along \mathcal{C} , see Fig. 1(b), together

with (24) leads to the particular form of Φ on \mathcal{A}^- . The representations (28), (29) describe the behaviour of ψ and χ in both sheets. Nevertheless, one may wish to consider the matrix Φ as a whole as a unique function of λ , which is therefore defined on both sheets of the K -surface. From this point of view, the equations (28), (29) describe Φ on one sheet only (say, on the upper sheet). Its values on the other (lower) sheet follow from (24).

Combining (28),(29) with (27) we obtain the axis values in the co-rotating system

$$\mathcal{A}^+ : \quad \Phi' = \left[\mathbf{1} + i(K - \zeta)\Omega_0 e^{-2U} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right] \times \left[\begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix} \right] \quad (30)$$

$$\mathcal{A}^- : \quad \Phi' = \left[\mathbf{1} + i(K - \zeta)\Omega_0 e^{-2U} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right] \times \left[\begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & G(K) \\ 0 & F(K) \end{pmatrix} \right], \quad (31)$$

where $\mathbf{1}$ is the 2×2 unit matrix. At the branch points $K_B = \zeta$ of K -surfaces belonging to axis values $\rho = 0, \zeta \in \mathcal{A}^\pm$, ψ and χ must be unique, i.e.

$$\mathcal{A}^+(K_B = \zeta) : \quad \Phi = \begin{pmatrix} \psi & \psi \\ \chi & -\chi \end{pmatrix} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(\zeta) & 0 \\ G(\zeta) & 1 \end{pmatrix} \quad (32)$$

$$\mathcal{A}^-(K_B = \zeta) : \quad \Phi = \begin{pmatrix} \psi & \psi \\ \chi & -\chi \end{pmatrix} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & G(\zeta) \\ 0 & F(\zeta) \end{pmatrix} \quad (33)$$

whence

$$\mathcal{A}^+ : \quad F(\zeta) = \frac{2}{f(\zeta) + \overline{f(\zeta)}}, \quad G(\zeta) = \frac{f(\zeta) - \overline{f(\zeta)}}{f(\zeta) + \overline{f(\zeta)}} \quad (34)$$

$$\mathcal{A}^- : \quad F(\zeta) = \frac{2f(\zeta)\overline{f(\zeta)}}{\overline{f(\zeta)} + f(\zeta)}, \quad G(\zeta) = \frac{\overline{f(\zeta)} - f(\zeta)}{f(\zeta) + \overline{f(\zeta)}} \quad (35)$$

Thus, $F(K)$ and $G(K)$ consist in a unique way of analytic continuations of the real and imaginary parts of the axis values of the Ernst potential $f(\zeta)$. Vice versa, $f(\zeta)$ follows from $F(K), G(K)$ for $K = \zeta$. Interestingly, the determinants of Φ and Φ' can be expressed in

terms of $\Re f$, $\Re f'$ and $F(K)$. From (17) ($\text{Tr } \Phi_{,z} \Phi^{-1} = (\ln \det \Phi)_{,z}$), (21) and (28)–(31), we have

$$\det \Phi = -2e^{2U} F(K), \quad \det \Phi' = -2e^{2V} F(K) \quad (36)$$

where $e^{2U} = \Re f$ and $e^{2V} = \Re f'$ ($U = U(\rho, \zeta)$, $V = V(\rho, \zeta)$).

We may now interpret the result of (30)–(35) of the integration of the LP along $\mathcal{A}^+ \mathcal{C} \mathcal{A}^-$: On the regular parts \mathcal{A}^\pm of the symmetry axis, Φ and Φ' can explicitly be expressed in terms of the axis values $f(\zeta)$ of the Ernst potential and its analytic continuations $F(K), G(K)$. To calculate $f(\zeta)$ one needs boundary values on \mathcal{B} . Accordingly, the integration along \mathcal{B} depends on the physical nature of the rotating body and can be performed in particular cases only. In the next section we will discuss black holes and rigidly rotating disks of dust.

B. Surface

1. One black hole

We identify the surface \mathcal{B} with the horizon \mathcal{H} . In Weyl coordinates, the event horizon \mathcal{H} of a single black hole covers the domain¹²,

$$\mathcal{H}: \quad \rho = 0, \quad K_1 \geq \zeta \geq K_2. \quad (37)$$

(In Fig. 1(a), the surface \mathcal{B} degenerates to a ‘straight line’ connecting the regular parts $\mathcal{A}^-, \mathcal{A}^+$ of the axis of symmetry.) Along \mathcal{H} , e^{2V} has to vanish, see (12),

$$\mathcal{H}: \quad e^{2V} \equiv (\xi^i + \Omega_0 \eta^i)(\xi_i + \Omega_0 \eta_i) = 0 \quad (\Omega_0 = \Omega_H). \quad (38)$$

Because of

$$e^{2V} = e^{2U} ([1 + \Omega_0 a]^2 - \Omega_0^2 \rho^2 e^{-4U}), \quad (39)$$

cf. (3), this implies

$$\mathcal{H}: \quad 1 + \Omega_0 a = 0. \quad (40)$$

Φ and Φ' can now be calculated along the horizon \mathcal{H} . From (17), (37),(38), (40) and (27) we obtain

$$\begin{aligned} \mathcal{H}: \quad \Phi &= \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}, \\ \Phi' &= 2i\Omega_0(K - \zeta) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}. \end{aligned} \quad (41)$$

The Ernst equations have to hold at K_1 and K_2 too. Hence, Φ and Φ' must be continuous in K_1 and K_2 . Considering (28)–(31) and (41), we are led to the conditions

$$\begin{aligned} \begin{pmatrix} f_1 & -1 \\ f_1 + 2i\Omega_0(K - K_1) & -1 \end{pmatrix} \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix} &= \\ \begin{pmatrix} f_1 & -1 \\ 2i\Omega_0(K - K_1) & 0 \end{pmatrix} \begin{pmatrix} U & V \\ W & X \end{pmatrix}, \end{aligned} \quad (42)$$

$$\begin{aligned} \begin{pmatrix} f_2 & -1 \\ f_2 + 2i\Omega_0(K - K_2) & -1 \end{pmatrix} \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix} &= \\ \begin{pmatrix} f_2 & -1 \\ 2i\Omega_0(K - K_2) & 0 \end{pmatrix} \begin{pmatrix} U & V \\ W & X \end{pmatrix}, \end{aligned}$$

where $f_1 = f(\zeta = K_1)$ and $f_2 = f(\zeta = K_2)$. Note that f_1 and f_2 are imaginary, see (36).

Eliminating the $UVWX$ matrix, we obtain

$$\mathcal{N} = \left(\mathbf{1} + \frac{\mathbf{F}_1}{2i\Omega_0(K - K_1)} \right) \left(\mathbf{1} + \frac{\mathbf{F}_2}{2i\Omega_0(K - K_2)} \right), \quad (43)$$

where

$$\mathbf{F}_1 = \begin{pmatrix} -f_1 & 1 \\ -f_1^2 & f_1 \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} f_2 & -1 \\ f_2^2 & -f_2 \end{pmatrix}, \quad (44)$$

$$\mathcal{N} = \begin{pmatrix} F & -G \\ G & (1 - G^2)/F \end{pmatrix}. \quad (45)$$

Obviously, the elements of \mathcal{N} are regular everywhere in the complex K -plane with the exception of the two simple poles at K_1 and K_2 ($\Im K_1 = 0 = \Im K_2$). The sum of the off-diagonal elements in (45) must be zero. This requirement leads to the constraints

$$f_1 = -f_2, \quad \Omega_0 = \frac{if_1(1 + f_1^2)}{(K_1 - K_2)(1 - f_1^2)}. \quad (46)$$

$F(K)$ and $G(K)$ take the form

$$F(K) = \frac{4\Omega_0^2(K^2 - K_1^2) + 4i\Omega_0 f_1 K - 2f_1^2}{4\Omega_0^2(K^2 - K_1^2)}, \quad G(K) = \frac{4i\Omega_0 K_1 + 2f_1}{4\Omega_0^2(K^2 - K_1^2)}. \quad (47)$$

Here we have chosen $K_1 = -K_2$, i.e., we have set the horizon in a symmetric position in the ρ, ζ -plane. Making use of (34) and (35) and eliminating Ω_0 by the second constraint equation we obtain the axis potential

$$\mathcal{A}^+ : \quad f = \frac{\zeta(1 + f_1^2) + (f_1^2 - 1 + 2f_1)K_1}{\zeta(1 + f_1^2) + (1 - f_1^2 + 2f_1)K_1}. \quad (48)$$

It can be useful to introduce the multipole moments mass M and angular momentum J by an asymptotic expansion of f ,

$$M = \frac{1 - f_1^2}{1 + f_1^2} K_1, \quad \frac{J}{M} = \alpha = \frac{2if_1 K_1}{1 + f_1^2} \quad (49)$$

and to replace f_1, K_1 in (47), (48) and (46):

$$F(K) = \frac{(K + M)^2 + \alpha^2}{K^2 + \alpha^2 - M^2}, \quad G(K) = \frac{2iM\alpha}{K^2 + \alpha^2 - M^2}. \quad (50)$$

To represent $f(\zeta)$, a simplifying parameterization is advisable,

$$f_1 = i \tan \varphi / 2, \quad \alpha = -M \sin \varphi, \quad K_1 = -K_2 = \sqrt{M^2 - \alpha^2} = M \cos \varphi, \quad \varphi = \bar{\varphi}. \quad (51)$$

This yields

$$\mathcal{A}^+ : \quad f = \frac{(\zeta - M) + iM \sin \varphi}{(\zeta + M) + iM \sin \varphi}. \quad (52)$$

Finally, the second constraint equation (46) becomes the well-known equation of state of black hole thermodynamics,

$$2M\Omega_0 = \frac{M}{\alpha} - \sqrt{\frac{M^2}{\alpha^2} - 1}, \quad (53)$$

connecting the angular velocity of the horizon with mass and angular momentum.

2. Two aligned black holes

The same procedure can be used to tackle the balance problem for two black holes. The question is whether the spin-spin repulsion of two aligned stationary black holes can compensate their gravitational attraction.

Here we have two horizons \mathcal{H}_1 and \mathcal{H}_2

$$\mathcal{H}_1 : \quad \rho = 0, \quad K_1 \geq \zeta \geq K_2, \quad \mathcal{H}_2 : \rho = 0, \quad K_3 \geq \zeta \geq K_4 \quad (54)$$

separated by a piece of the regular symmetry axis \mathcal{A}^0

$$\mathcal{A}^0 : \quad K_2 \geq \zeta \geq K_3. \quad (55)$$

As a characteristic black hole property, the norm of the Killing vectors of the co-rotating frameworks has to vanish along the horizons,

$$\mathcal{H}_1 : \quad (\xi_i + \Omega_0^1 \eta_i)(\xi^i + \Omega_0^1 \eta^i) = 0, \quad \mathcal{H}_2 : \quad (\xi_i + \Omega_0^2 \eta_i)(\xi^i + \Omega_0^2 \eta^i) = 0, \quad (56)$$

where Ω_0^1, Ω_0^2 are the constant angular velocities of the respective horizons.

Following the arguments for one black hole, we arrive at¹³

$$\mathcal{N} = \prod_{i=1}^4 \left(1 + \frac{\mathbf{F}_i}{2i\Omega_i(K - K_i)} \right) \quad (57)$$

where $\Omega_1 = \Omega_2 = \Omega_0^2$, $\Omega_3 = \Omega_4 = \Omega_0^1$ and

$$\mathbf{F}_i = (-1)^i \begin{pmatrix} f_i & -1 \\ f_i^2 & -f_i \end{pmatrix}, \quad (58)$$

whence

$$\begin{aligned} F(K) &= \frac{p_4(K)}{(K - K_1)(K - K_2)(K - K_3)(K - K_4)} \\ G(K) &= \frac{p_2(K)}{(K - K_1)(K - K_2)(K - K_3)(K - K_4)}, \end{aligned} \quad (59)$$

where $p_4(K)$ and $p_2(K)$ are polynomials in K of the indicated orders. From (59) together with (32), (33) we may read off the axis values of the Ernst potential. For the upper axis we obtain the structure

$$\mathcal{A}^+ : \quad f(\zeta) = \frac{q_2(\zeta)}{Q_2(\zeta)}. \quad (60)$$

We need not use the representation of the explicit form of the second order polynomials $q_2(\zeta), Q_2(\zeta)$ and of the constraints resulting from $G = \mathcal{N}_{21} = -\mathcal{N}_{12}$. Namely, from the fact that $f(\zeta)$ is a quotient of polynomials of the same (even) order, it is clear that the desired two black hole solution can be generated by a Bäcklund transformation (in our case by a two-fold Bäcklund transformation) from the Minkowski space. (Note that the axis values of the Ernst potential determine solutions of the Ernst equation in a unique way.)

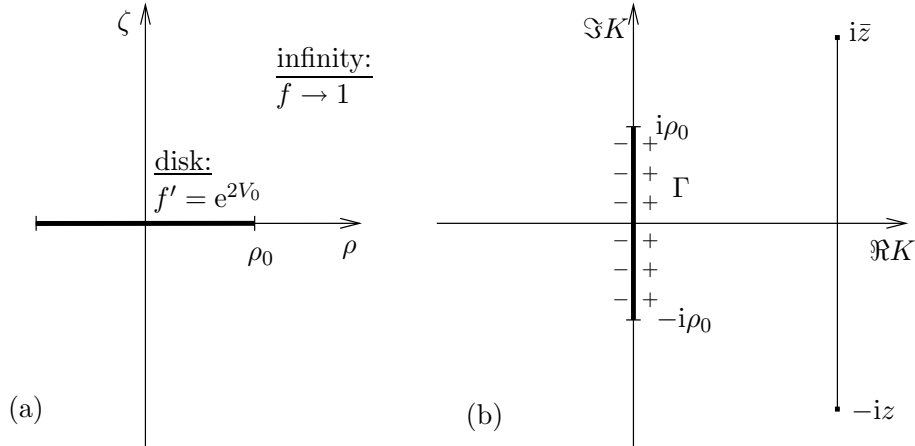


FIG. 2: (a) Boundary value problem for the rotating disk of dust, (b) 2-sheeted K -surface of the rotating disk with branch points at $-iz, i\bar{z}$.²

The four constraints $\mathcal{N}_{21} = -\mathcal{N}_{12}$ ensure that the constants K_i , $f_i = -\bar{f}_i$ ($i = 1, \dots, 4$), Ω_0^1, Ω_0^2 may be expressed by two position parameters and the masses and angular momenta of the two black holes.

The Bäcklund generated solution belonging to (60) known as the “double Kerr solution” was intensively discussed by several authors^{14,15}. It turned out that there are necessarily struts between the “horizons”. Since we have shown, by solving the boundary value problem, that Bäcklund generated solutions are the *only* candidates to describe aligned balanced black holes, we may now assert that black holes cannot be balanced at all.

3. Rigidly rotating disks of dust

Disks of dust can be considered to be extremely flattened spheroids (Fig. 2(a)) consisting of perfect fluid matter³. One can show that, for rigidly rotating dust, the boundary conditions (4) and (11) have to be complemented by the condition $b' = \Im f' = 0$ on the disk (\mathcal{B}). This condition follows from the Einstein equations as a transition condition from a divergence-free part of those equations via Gauss’s theorem³. Thus we have to take into consideration

$$\mathcal{A}^\pm : \text{regularity of } f, \quad \mathcal{B} : f' = e^{2V_0}, \quad \mathcal{C} : f \rightarrow 1, \quad (61)$$

see Fig. 2(a). On the disk, the Linear Problem of the co-rotating system of reference takes the form

$$\mathcal{B}: \quad \Phi'_{,\rho} = -\frac{\rho}{\sqrt{K^2 + \rho^2(f' + \bar{f}')}} \begin{pmatrix} 0 & \bar{f}'_{,\zeta} \\ f'_{,\zeta} & 0 \end{pmatrix} \Phi', \quad (62)$$

where Φ' and f' are the ‘co-rotating’ Φ -matrix and the ‘co-rotating’ Ernst potential on the disk. This relation must be discussed under the ‘boundary conditions’

$$\Phi'(\rho = 0, \zeta = 0^+, \lambda)|_B = \Phi'(\rho = 0, \zeta = 0^+, \lambda)|_{\mathcal{A}^+},$$

cf. (30) and

$$\Phi'(\rho = 0, \zeta = 0^-, \lambda)|_B = \Phi'(\rho = 0, \zeta = 0^-, \lambda)|_{\mathcal{A}^-},$$

cf. (31).

Again, this discussion allows the construction of $F(K)$ and $G(K)$ and, via (34) and (35), the construction of the axis values $f(\zeta)$ of the Ernst potential. We first take advantage of the symmetry of the problem which implies $\overline{f(\rho, \zeta)} = f(\rho, -\zeta)$ and connects the ζ -derivatives of f' above ($\zeta = 0^+$) and below ($\zeta = 0^-$) the disk

$$\mathcal{B}: \quad \bar{f}'_{,\zeta}|_{\zeta=0^+} = -f'_{,\zeta}|_{\zeta=0^-}. \quad (63)$$

As a consequence, the LP (62) connects the matrix $\overset{\text{A}}{\Phi}$ above the disk, $\overset{\text{A}}{\Phi} = \Phi'(\rho, \zeta = 0^+, K)$, with the matrix $\overset{\text{B}}{\Phi}$ below the disk, $\overset{\text{B}}{\Phi} = \Phi'(\rho, \zeta = 0^-, K)$,

$$\mathcal{B}: \quad \overset{\text{A}}{\Phi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overset{\text{B}}{\Phi} \mathbf{H}(K), \quad (64)$$

where the matrix $\mathbf{H}(K)$ (the “integration constant”) does not depend on $\rho \in \mathcal{B}$. At the rim of the disk we have

$$\overset{\text{A}}{\Phi}(\rho_0, 0, K) = \overset{\text{B}}{\Phi}(\rho_0, 0, K) = \overset{\text{r}}{\Phi}. \quad (65)$$

Because of (64), the rim matrix $\overset{\text{r}}{\Phi}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overset{\text{r}}{\Phi}$ can be expressed in terms of

$\overset{\text{A}}{\Phi} = \Phi'(\rho, \zeta = 0^+, K)$, $\overset{\text{B}}{\Phi} = \Phi'(\rho, \zeta = 0^-, K)$. Note that Φ is considered to be a holomorphic function of λ and therefore a function living on the 2-sheeted Riemann K -surface of Fig. 1(b). Hence we have to discuss the rim matrix as a function of K on both sheets.

Any Φ multiplied from the right by a matrix function of K is again a solution of the LP. The discussion of the rim matrix simplifies after the following redefinition

$$\mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix} \Phi^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}. \quad (66)$$

Using (30)(31) we obtain

$$\mathcal{R} = \begin{cases} e^{2V_0} \mathcal{M} \mathcal{S}^{-1} & \text{on the upper sheet} \\ -e^{2V_0} \mathcal{S}^{-1} \mathcal{M} & \text{on the lower sheet} \end{cases} \quad (67)$$

where

$$\mathcal{M} = \begin{pmatrix} G(K) & (G^2 - 1)/F \\ -F(K) & -G(K) \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} f_0 \overline{f_0} - 4\Omega_0^2 K^2 & ib_0 + 2i\Omega_0 K \\ ib_0 - 2i\Omega_0 K & -1 \end{pmatrix} \quad (68)$$

and

$$f_0 = e^{2V_0} + ib_0 = f(\zeta = 0^+). \quad (69)$$

Note that $\mathcal{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{N} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Obviously, $\text{Tr } \mathcal{R} = \text{Tr } \mathcal{R}^{-1} = 0$ and $\mathcal{M}^2 = \mathbf{1}$, whence

$$\text{Tr } \mathcal{M} \mathcal{S}^{-1} = \text{Tr } \mathcal{S} \mathcal{M} = 0. \quad (70)$$

This relation interlinks $F(K)$ and $G(K)$ and, because of (34),(35) real and imaginary part of the axis values $f(\zeta)$ of the Ernst potential^{16,17}.

We next wish to determine $F(K)$ and $G(K)$ which in turn determine $f(\zeta)$. To this end we consider $\Phi(\rho, \zeta, \lambda)$, for fixed coordinates ρ, ζ as a function of λ . We have already used the initial conditions $\psi = \chi = 1$ for some $\rho = \rho_0 = 0$, $\zeta = \zeta_0 \in \mathcal{A}^+$ prescribed in one sheet ($\lambda = -1$) of the K -plane. In principle the behaviour of Φ in the other sheet and at all points in the ρ, ζ -plane can be calculated by integrating the LP along a suitable path. However, the coefficients $A(\rho, \zeta)$, $B(\rho, \zeta)$ in the LP (17) are not explicitly known. Nevertheless, their regular behaviour outside the disk together with the boundary values on the disk, cf. (61), provides us with defining properties for Φ . One of them may be taken from Fig. 1(b): Since the domain of the disk, $0 \leq \rho \leq \rho_0$, $\zeta = 0^\pm$ is a non-vacuum domain, where the LP fails, Φ at the branch point pairs $K = i\rho + 0^\pm, -i\rho + 0^\pm$, $0 \leq \rho \leq \rho_0$ cannot “pass” through

the contour $\Gamma : -\rho_0 \leq \Im K \leq \rho_0$, i.e. Φ has a well-defined jump between opposite points along the contour Γ , see Fig. 2. A careful discussion would show that $\Phi(\rho, \zeta, \lambda)$, for fixed coordinate values ρ, ζ outside the disk ($\rho, \zeta \notin \mathcal{B}$), is a regular function in λ outside Γ and jumps along Γ , i.e. Φ satisfies a (regular) Riemann-Hilbert problem.

Consider now the jump $\Phi_+^{-1}\Phi_-$, where the signs indicate the two sides of Γ , cf. Fig. 2(b). The LP tells us that $\Phi_+^{-1}\Phi_-$ does not depend on the coordinates and is therefore a function \mathcal{D} of the contour alone,

$$\begin{aligned}\Phi_+^{-1}\Phi_- &= \mathcal{D}_u(K), & K \in \Gamma_u \\ \Phi_+^{-1}\Phi_- &= \mathcal{D}_l(K), & K \in \Gamma_l,\end{aligned}\tag{71}$$

where u marks the upper and l the lower sheet. Since the jump contours Γ_u, Γ_l and the jump matrices $\mathcal{D}_u, \mathcal{D}_l$ are the same for all values of ρ, ζ (i.e. for all Riemann surfaces with different branch points), we may express \mathcal{D}_u and \mathcal{D}_l in terms of the axis values of Φ ,

$$\mathcal{D}_u(K) = \begin{pmatrix} F_+ & 0 \\ G_+ & 1 \end{pmatrix}^{-1} \begin{pmatrix} F_- & 0 \\ G_- & 1 \end{pmatrix}, \quad K \in \Gamma_u. \tag{72}$$

A similar relation for \mathcal{D}_l may be obtained via (24). As a consequence of (71), (72) the matrix $\Phi \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1}$ does not jump along Γ_u . Because of (27) the same holds for $\Phi' \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1}$ and, finally, for \mathcal{R} as defined in (66). Consider now the Riemann K -surface of the disk rim $\rho = \rho_0$, $\zeta = 0$. The cut between the branch points $K_B = \pm i\rho_0$ coincides with the contour Γ_u, Γ_l which are on the two “bridges” connecting crosswise the upper with the lower sheet. Since \mathcal{R} does not jump on Γ_u , we have, according to (67) $(\mathcal{M}\mathcal{S}^{-1})_- = -(\mathcal{S}^{-1}\mathcal{M})_+$. Though \mathcal{R} does not jump, F and G do jump, cf. (72). Note that $F(K)$ and $G(K)$ are unique functions of K . Hence, there is only one contour $\Gamma : \Re K = 0, -\rho_0 \leq \Im K \leq \rho_0$ where \mathcal{M} (with the elements $F(K), G(K)$) does jump. Since Φ is analytic outside Γ_u, Γ_l , the matrix \mathcal{M} must be analytic outside Γ . Thus we obtain $F(K)$ and $G(K)$ from the Riemann-Hilbert problem

$$\begin{aligned}K \in \Gamma : & \quad \mathcal{S}\mathcal{M}_- = -\mathcal{M}_+\mathcal{S} \\ K \notin \Gamma : & \quad \mathcal{M}(K) \text{ analytic in } K,\end{aligned}\tag{73}$$

\mathcal{S} and \mathcal{M} as in (68). (Note that the elements of \mathcal{S} , which are polynomials and the elements of \mathcal{S}^{-1} which are rational functions in K do not jump along Γ .) There is no jump at the end

points of the contour $K = \pm i\rho_0$, $\mathcal{M}(\pm i\rho_0)_- = \mathcal{M}(\pm i\rho_0)_+$. As a consequence, one obtains $\text{Tr } \mathcal{S}(\pm i\rho_0) = 0$, i.e., the parameter relation

$$f_0 \bar{f}_0 + 4\Omega_0^2 \rho_0^2 = 1. \quad (74)$$

It turns out that the Riemann-Hilbert problem (73) has a unique solution $\mathcal{M}(K)$ in the parameter region

$$0 \leq \mu = 2\Omega_0^2 e^{-2V_0} \rho_0^2 < \mu_0 = 4.62966184\dots \quad (75)$$

An important step on the way to this solution is the diagonalization of \mathcal{S} . Finally, one obtains $F(K)$, $G(K)$ and the axis values of the Ernst potential $f(\zeta)$ in terms of elliptic theta functions. We need not go this road. As we shall see in the next section, we can use the Riemann-Hilbert problem (73) to formulate a more general Riemann-Hilbert problem which will yield the complete disk of dust solution in terms of hyperelliptic theta functions.

VI. ERNST POTENTIAL EVERYWHERE

A. Kerr solution

In the preceding section, we analyzed the axis values of the Ernst potential. We will now construct the complete solutions $f(\rho, \zeta)$ of our boundary value problems from the information about the behaviour along the axis of symmetry gained by the discussion of the direct problem.

There is, of course, no question that the discussion of the black hole case in VB1 will lead to the famous Kerr solution (in Weyl coordinates (1)). The point made here is that this solution describing the stationary rotating black hole can be derived from a boundary value problem.

To achieve our goal it is useful to exploit the gauge freedom of multiplying Φ from the right by an arbitrary matrix function of K . The transformation

$$\tilde{\Phi} = \frac{K^2 - \alpha^2 - M^2}{K[(K + M)^2 + \alpha^2]} \Phi \begin{pmatrix} K + m & i\alpha \\ i\alpha & K + m \end{pmatrix} \quad (76)$$

preserves the properties (22)–(25) and enables the calculation of f via (26). Because of (36), (50), (28) and (18), the determinant of $\tilde{\Phi}$ becomes

$$\det \tilde{\Phi} = \gamma \frac{(K + iz)^2}{K^2} (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2), \quad \gamma = \gamma(\rho, \zeta) = -\frac{2e^{2U(\rho, \zeta)}}{(1 - \lambda_1^2)(1 - \lambda_2^2)}, \quad (77)$$

where

$$\lambda_i^2 = \frac{K_i - i\bar{z}}{K_i + iz} \quad (i = 1, 2). \quad (78)$$

This form of the determinant together with the axis values of $\tilde{\Phi}$ tells us that $\tilde{\Phi}$ must be a quadratic matrix polynomial of λ ,

$$\tilde{\Phi} = \frac{K + iz}{K}(\mathbf{C} + \mathbf{D}\lambda + \mathbf{E}\lambda^2), \quad (79)$$

where the 2×2 matrices \mathbf{C} , \mathbf{D} , \mathbf{E} are functions of ρ , ζ alone. It can be shown¹⁸ that $\tilde{\Phi}$ with (77) satisfies the LP. (It is a Bäcklund transformation of the trivial solution $f = 1$.)

According to (77), $\tilde{\Phi}(\rho, \zeta, \lambda_i)$ ($i = 1, 2$) must have a null eigenvector b_i in the zeros λ_i ,

$$\tilde{\Phi}(\rho, \zeta, \lambda_i)b_i = 0 \quad (i = 1, 2). \quad (80)$$

From the LP it follows that the elements of b_i have to be constants. Hence, the quotient

$$\frac{\tilde{\chi}(\rho, \zeta, \lambda)}{\tilde{\chi}(\rho, \zeta, -\lambda)} = -\frac{C_{21} + D_{21}\lambda + E_{21}\lambda^2}{C_{21} - D_{21}\lambda + E_{21}\lambda^2}, \quad (81)$$

where the coefficients are elements of the matrices \mathbf{C} , \mathbf{D} , \mathbf{E} must be a constant at $\lambda = \lambda_i$ ($i = 1, 2$). The values of the two constants ($i = 1, 2$) can be read off from the axis values of $\tilde{\Phi}$ resulting from (81) together with (28), (29), (50), (51) and (76),

$$\frac{\tilde{\chi}(\lambda_1)}{\tilde{\chi}(-\lambda_1)} = -i \cot \frac{\varphi}{2}, \quad \frac{\tilde{\chi}(\lambda_2)}{\tilde{\chi}(-\lambda_2)} = i \cot \frac{\varphi}{2}. \quad (82)$$

Note that $\tilde{\chi}(-1) = 1$ implies

$$C_{21} - D_{21} + E_{21} = 1. \quad (83)$$

These three conditions fix the coefficients C_{21} , D_{21} , E_{21} via a linear algebraic system. Finally, we obtain the Ernst potential everywhere from $f = \tilde{\chi}(1)/\tilde{\chi}(-1)$,

$$f(\rho, \zeta) = \frac{r_1 e^{i\varphi} + r_2 e^{i\varphi} - 2M \cos \varphi}{r_1 e^{i\varphi} + r_2 e^{i\varphi} + 2M \cos \varphi}, \quad (84)$$

where

$$r_i^2 = (K_i - \zeta)^2 + \rho^2 \quad (i = 1, 2)$$

with $K_1 = -K_2$ and φ as in (51). This is the Ernst potential f of the Kerr solution in Weyl-Papapetrou coordinates. By virtue of (15) and (16), this potential determines all metric coefficients in the line element (1).

B. Disk of dust solution

In order to construct the Φ -matrix for arbitrary values of ρ , ζ and λ , let us return to the Riemann-Hilbert problem (73). As we have seen, the matrix $\Phi \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1}$ does not jump along Γ_u . Analogously, $\Phi \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix}^{-1}$ does not jump along Γ_l . The images Γ_λ of Γ_u and $\Gamma_{-\lambda}$ of Γ_l inherit these properties, which are essential to the following deductions.

To formulate a Riemann-Hilbert problem in the λ -plane, we define two matrices,

$$\begin{aligned} \mathcal{L} &:= \Phi \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{M} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix} \Phi^{-1} \\ &= \Phi \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix} \Phi^{-1} \\ &= \Phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi^{-1}, \end{aligned} \quad (85)$$

$$\begin{aligned} \mathcal{Q} &:= e^{-2V_0} \Phi \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathcal{S} + w\mathbf{1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix} \Phi^{-1} \\ &= e^{-2V_0} \Phi \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathcal{S} + w\mathbf{1}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F & 0 \\ G & 1 \end{pmatrix} \Phi^{-1}, \end{aligned} \quad (86)$$

where

$$w = -\frac{1}{2} \text{Tr } \mathcal{S} = 2\Omega_0^2(K^2 + \rho_0^2).$$

Here we have made use of the parameter relation (74). Since \mathcal{S} and w are polynomials in K and therefore rational functions in λ , the matrix \mathcal{Q} has no jump at all. Taking the asymptotics of \mathcal{S} and w into account, \mathcal{Q} must take the following polynomial structure in λ

$$\mathcal{Q} = (K + iz)^2 \begin{pmatrix} q_1 & q_2 \\ q_3 & -q_2 \end{pmatrix}, \quad q_1 = k\lambda + l\lambda^3, \quad q_2 = m + n\lambda^2 + p\lambda^4, \quad q_3 = q + r\lambda^2 + s\lambda^4, \quad (87)$$

where k, l, m, n, p, q, r, s are functions of ρ, ζ alone. From the definitions (85), (86) and the condition (70), we may derive

$$\mathcal{Q}\mathcal{L} = -\mathcal{L}\mathcal{Q} \quad (88)$$

whereas the particular Riemann-Hilbert problem (73) has the continuation

$$\begin{aligned} \lambda \in \Gamma_\lambda : \quad & (\mathcal{Q} + e^{-2V_0} w \mathbf{1}) \mathcal{L}_- = -\mathcal{L}_+ (\mathcal{Q} + e^{-2V_0} w \mathbf{1}) \\ \lambda \in \Gamma_{-\lambda} : \quad & (\mathcal{Q} - e^{-2V_0} w \mathbf{1}) \mathcal{L}_- = -\mathcal{L}_+ (\mathcal{Q} - e^{-2V_0} w \mathbf{1}) \\ \lambda \notin \Gamma_\lambda, \Gamma_{-\lambda} : \quad & \mathcal{L} \text{ analytic in } \lambda \end{aligned} \quad (89)$$

The following solution of the regular Riemann-Hilbert problem (89) is based on the diagonalization of \mathcal{Q} .

We consider a function Ψ defined by

$$\Psi := \frac{1}{\sqrt{w^2 + e^{4V_0}}} \ln \frac{\hat{\mathcal{L}}_{22} + \sqrt{1 + w^2 e^{-4V_0}} \hat{\mathcal{L}}_{21}}{\hat{\mathcal{L}}_{22} - \sqrt{1 + w^2 e^{-4V_0}} \hat{\mathcal{L}}_{21}}, \quad (90)$$

where

$$\hat{\mathcal{L}} = \mathcal{L} \begin{pmatrix} 1 & \mathcal{Q}_{11} \\ 0 & \mathcal{Q}_{21} \end{pmatrix}. \quad (91)$$

Note that Ψ has no branch points at the zeroes $K_1, K_2, \bar{K}_1 = -K_2$ and $\bar{K}_2 = -K_1$ of $w^2 + e^{4V_0}$,

$$K_1^2 = \rho_0^2 \frac{i - \mu}{\mu}, \quad K_2^2 = \rho_0^2 \frac{i + \mu}{\mu} \quad (\Re K_1 < 0, \quad \Re K_2 > 0, \quad \mu \text{ as in (75)}) \quad (92)$$

(Ψ is unaffected by a change in the sign of $\sqrt{w^2 + e^{4V_0}}$). It is an odd function of λ (vanishing at $\lambda = 0$ and at $\lambda = \infty$). Therefore, the function

$$\hat{\Psi} = \Psi / [\lambda(K + iz)] = \Psi / \sqrt{(K - i\bar{z})(K + iz)} \quad (93)$$

can be discussed as a unique function of K with the following properties:

(i) Along Γ , because of (89), it jumps according to

$$\hat{\Psi}_- = \hat{\Psi}_+ + \frac{2}{\sqrt{(K - i\bar{z})(K + iz)} \sqrt{w^2 + e^{4V_0}}} \ln \frac{\sqrt{w^2 + e^{4V_0}} + w}{\sqrt{w^2 + e^{4V_0}} - w}, \quad (94)$$

(ii) Because of

$$\hat{\mathcal{L}}_{21}^2 (1 + w^2 e^{-4V_0}) - \hat{\mathcal{L}}_{22}^2 = \mathcal{Q}_{21}^2, \quad (95)$$

$$\mathcal{Q}_{21} = -\frac{2f\Omega_0^2 e^{-2V_0}}{f + \bar{f}} (K - K_a)(K - K_b), \quad (96)$$

the behaviour for $K \rightarrow K_{a/b}$ is given by

$$\hat{\Psi} \rightarrow \frac{\pm 2}{\sqrt{(K_{a/b} - i\bar{z})(K_{a/b} + iz)(w_{a/b}^2 + e^{4V_0})}} \ln(K - K_{a/b}) \quad \text{as } K \rightarrow K_{a/b}. \quad (97)$$

(The ambiguity of sign can be compensated for by the square root.)

(iii) The behaviour for $K \rightarrow \infty$, because of the definitions of \mathcal{Q} and \mathcal{L} , is given by

$$\hat{\Psi} \rightarrow \frac{\ln f}{\Omega^2 K^3} \quad \text{as} \quad K \rightarrow \infty. \quad (98)$$

These properties are realized by the following representation of $\hat{\Psi}$:

$$\begin{aligned} \hat{\Psi} = & \frac{1}{\pi i} \int_{-i\rho_0}^{i\rho_0} \frac{\ln \frac{\sqrt{w'^2 + e^{4V_0}} + w'}{\sqrt{w'^2 + e^{4V_0}} - w'}}{\sqrt{(K' - i\bar{z})(K' + iz)} \sqrt{w'^2 + e^{4V_0}} (K' - K)} dK' \\ & - 2 \int_{K_1}^{K_a} \frac{1}{\sqrt{(K' - i\bar{z})(K' + iz)} (w'^2 + e^{4V_0}) (K' - K)} dK' \\ & - 2 \int_{K_2}^{K_b} \frac{1}{\sqrt{(K' - i\bar{z})(K' + iz)} (w'^2 + e^{4V_0}) (K' - K)} dK', \end{aligned} \quad (99)$$

where K_a and K_b have to be determined such that $\hat{\Psi} = \mathcal{O}(K^{-3})$. The lower limits of integration in the last two integrals have been fixed to obtain the correct result in the Newtonian limit $\mu \rightarrow 0$ where $K_a/K_1 = 1 + \mathcal{O}(\mu^2)$ and $K_b/K_2 = 1 + \mathcal{O}(\mu^2)$. (A systematic post-Newtonian expansion of the solution is given in¹⁹.) Note that the last two terms in Eq. (99) may also be interpreted as follows,

$$2 \left(\int_{K_1}^{K_a} + \int_{K_2}^{K_b} \right) = 2 \left(\int_{K_a}^{K_1} \{-\} + \int_{K_2}^{K_b} \{-\} \right) = \int_{K_a}^{K_b} \{1\} + \int_{K_a}^{K_b} \{2\}, \quad (100)$$

showing that nothing special happens at K_1 and K_2 . In this symbolic notation $\{-\}$ indicates that the square root is meant to have the opposite sign with reference to the first term; $\{1\}$ and $\{2\}$ denote different paths in the complex K -plane, which are chosen such that the closed integral

$$\oint = \int_{K_a}^{K_b} \{1\} - \int_{K_a}^{K_b} \{2\} = 2 \int_{K_1}^{K_2} \quad (101)$$

is performed around a contour enclosing the branch points K_1 and K_2 of $\sqrt{w^2 + e^{4V_0}}$. In the subsequent formulae we normalize K and introduce

$$X = \frac{K}{\rho_0}, \quad X_{a/b} = \frac{K_{a/b}}{\rho_0}, \quad X_{1/2} = \frac{K_{1/2}}{\rho_0}. \quad (102)$$

An asymptotic expansion of Eq. (99) for $X \rightarrow \infty$ ($K \rightarrow \infty$) leads, according to (98), to

$$\ln f = \mu \left[\int_{X_1}^{X_a} \frac{X^2 dX}{W} + \int_{X_2}^{X_b} \frac{X^2 dX}{W} - \int_{-i}^i \frac{hX^2 dX}{W_1} \right], \quad (103)$$

$$\int_{X_1}^{X_a} \frac{dX}{W} + \int_{X_2}^{X_b} \frac{dX}{W} = \int_{-i}^i \frac{h dX}{W_1}, \quad \int_{X_1}^{X_a} \frac{X dX}{W} + \int_{X_2}^{X_b} \frac{X dX}{W} = \int_{-i}^i \frac{hX dX}{W_1}, \quad (104)$$

where the lower integration limits X_1, X_2 are given by

$$X_1^2 = \frac{i - \mu}{\mu}, \quad X_2^2 = -\frac{i + \mu}{\mu} \quad (\Re X_1 < 0, \quad \Re X_2 > 0), \quad (105)$$

whereas the upper limits X_a, X_b must be calculated from the integral equations (104). Here we have introduced the abbreviations

$$W = W_1 W_2, \quad W_1 = \sqrt{(X - \zeta/\rho_0)^2 + (\rho/\rho_0)^2}, \quad W_2 = \sqrt{1 + \mu^2(1 + X^2)^2} \quad (106)$$

and

$$h = \frac{\ln \left(\sqrt{1 + \mu^2(1 + X^2)^2} + \mu(1 + X^2) \right)}{\pi i \sqrt{1 + \mu^2(1 + X^2)^2}}. \quad (107)$$

The third integral in (103) as well as the integrals on the right-hand sides in (104) have to be taken along the imaginary axis in the complex X -plane with h and W_1 fixed according to $\Re W_1 < 0$ (for ρ, ζ outside the disk) and $\Re h = 0$. The task of calculating the upper limits X_a, X_b in (104) from

$$u = \int_{-i}^i \frac{h dX}{W_1}, \quad v = \int_{-i}^i \frac{hX dX}{W_1} \quad (108)$$

is known as Jacobi's inversion problem. Göpel²⁰ and Rosenhain²¹ were able to express the hyperelliptic functions $X_a(u, v)$ and $X_b(u, v)$ in terms of (hyperelliptic) theta functions. Later on it turned out that even the first two integrals in (103) can be expressed by theta functions in u and v ! A detailed introduction into the related mathematical theory which was founded by Riemann and Weierstraß may be found in^{22,23,24}. The representation of the Ernst potential (103) in terms of theta functions can be taken from Stahl's book, see²², page 311, Eq. (5). Here is the result: Defining a theta function $\vartheta(x, y; p, q, \alpha)$ by

$$\vartheta(x, y; p, q, \alpha) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} p^{m^2} q^{n^2} e^{2mx + 2ny + 4mn\alpha} \quad (109)$$

one can reformulate the expressions (103), (104) to give

$$f = \frac{\vartheta(\alpha_0 u + \alpha_1 v - C_1, \beta_0 u + \beta_1 v - C_2; p, q, \alpha)}{\vartheta(\alpha_0 u + \alpha_1 v + C_1, \beta_0 u + \beta_1 v + C_2; p, q, \alpha)} e^{-(\gamma_0 u + \gamma_1 v + \mu w)} \quad (110)$$

with u and v as in (108) and

$$w = \int_{-i}^i \frac{hX^2 dX}{W_1}. \quad (111)$$

The normalization parameters $\alpha_0, \alpha_1; \beta_0, \beta_1; \gamma_0, \gamma_1$, the moduli p, q, α of the theta function and the quantities C_1, C_2 are defined on the two sheets of the hyperelliptic Riemann surface related to

$$W = \mu \sqrt{(X - X_1)(X - \bar{X}_1)(X - X_2)(X - \bar{X}_2)(X - i\bar{z}/\rho_0)(X + iz/\rho_0)}, \quad (112)$$

see Figure 3. There are two normalized Abelian differentials of the first kind

$$d\omega_1 = \alpha_0 \frac{dX}{W} + \alpha_1 \frac{XdX}{W} \quad (113)$$

$$d\omega_2 = \beta_0 \frac{dX}{W} + \beta_1 \frac{XdX}{W} \quad (114)$$

defined by

$$\oint_{a_m} d\omega_n = \pi i \delta_{mn} \quad (m = 1, 2; n = 1, 2). \quad (115)$$

Eq. (115) consists of four linear algebraic equations and yields the four parameters $\alpha_0, \alpha_1, \beta_0, \beta_1$ in terms of integrals extending over the closed (deformable) curves a_1, a_2 . It can be shown that there is one normalized Abelian differential of the third kind

$$d\omega = \gamma_0 \frac{dX}{W} + \gamma_1 \frac{XdX}{W} + \mu \frac{X^2 dX}{W} \quad (116)$$

with vanishing a -periods,

$$\oint_{a_j} d\omega = 0 \quad (j = 1, 2). \quad (117)$$

This equation defines γ_0, γ_1 (again via a linear algebraic system). The Riemann matrix

$$(B_{ij}) = \begin{pmatrix} \ln p & 2\alpha \\ 2\alpha & \ln q \end{pmatrix} \quad (i = 1, 2; j = 1, 2) \quad (118)$$

(with negative definite real part) is given by

$$B_{ij} = \oint_{b_i} d\omega_j \quad (119)$$

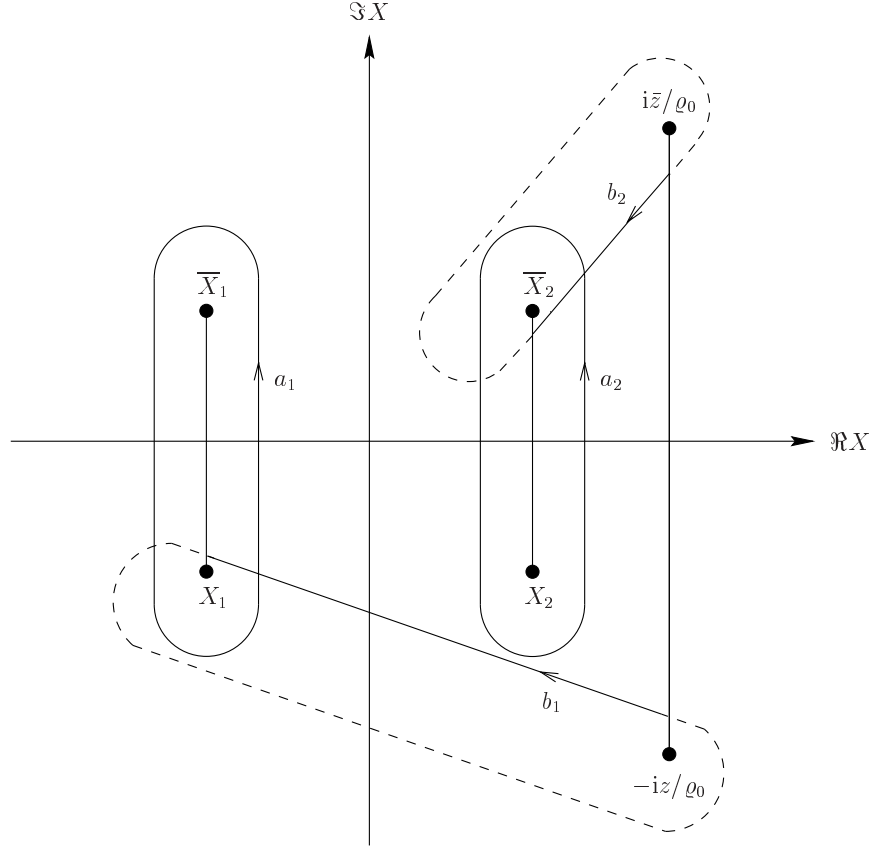


FIG. 3: Riemann surface with cuts between the branch points X_1 and \bar{X}_1 , X_2 and \bar{X}_2 , $-iz/\rho_0$ and $i\bar{z}/\rho_0$. Also shown are the four periods a_i and b_i ($i = 1, 2$). (Continuous/dashed lines belong to the upper/lower sheet defined by $W \rightarrow \pm\mu X^3$ as $X \rightarrow \infty$.)³

and defines the moduli p , q , α of the theta function (109). Finally, the quantities C_1 , C_2 can be calculated by

$$C_i = - \int_{-iz/\rho_0}^{\infty^+} d\omega_i \quad (i = 1, 2), \quad (120)$$

where $+$ denotes the upper sheet. Obviously, all the quantities entering the theta functions and the exponential function in (110) can be expressed in terms of well-defined integrals and depend on the three parameters ρ/ρ_0 , ζ/ρ_0 , μ . The corresponding “tables” for α_i , β_i , γ_i , C_i , B_{ij} , u , v , w can easily be calculated by numerical integrations. Fortunately, theta series like (109) converge rapidly. For $0 < \mu < \mu_0$, the solution (110) is analytic *everywhere* outside the disk — even at the rings $-iz/\rho_0 = X_1, X_2$. The complete metric, calculated according to (1) and (14)–(16) is given in the Appendix.

In the framework of the completely integrable evolution equations, the solution (110)

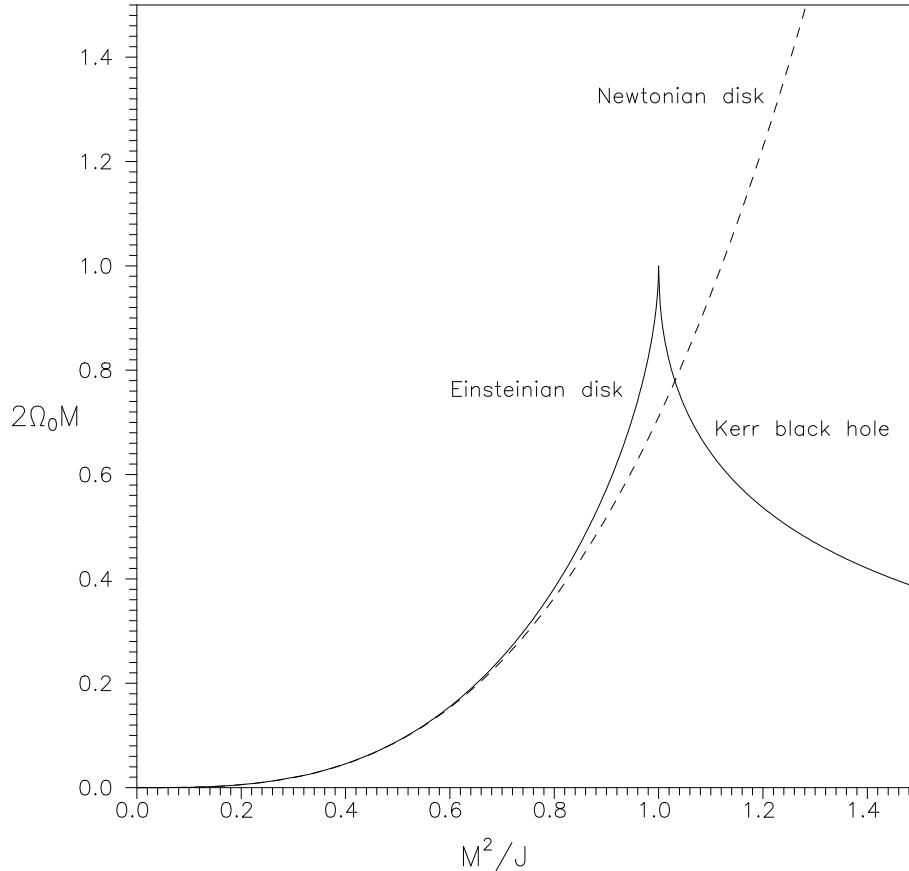


FIG. 4: Relation between $2\Omega_0 M$ and M^2/J for the classical Maclaurin disk (dashed line), the general-relativistic dust disk and the Kerr black-hole¹⁷

may be interpreted as a ‘Bäcklund-like’ transformation of well-defined ‘seed’ solutions u, v, w satisfying axisymmetric Laplace equations. The transformation ‘parameters’ $\alpha_0, \beta_0; \alpha_1, \beta_1; \gamma_0, \gamma_1; p, q, \alpha; C_1, C_2$ depend on the 6 branch points of the 2-sheeted Riemann K -surface associated with the function $W = W(X)$, cf. (106), and do not depend on u, v, w . All in all, f is a function of the 2 parameters ρ_0 and μ and the 2 cylindrical coordinates ρ and ζ . For $\mu \ll 1$ we obtain the Maclaurin disk as the Newtonian limit.

VII. PHYSICAL DISCUSSION

Since the Kerr black hole is also a 2 parameter solution it might be interesting to compare the behaviour of both solutions in dependence on common parameters, say, on mass M and angular momentum J . It must be possible to express the area of the horizon and the disk, the radius ρ_0 of the disk or other physical quantities in terms of M and J . A very illustrative

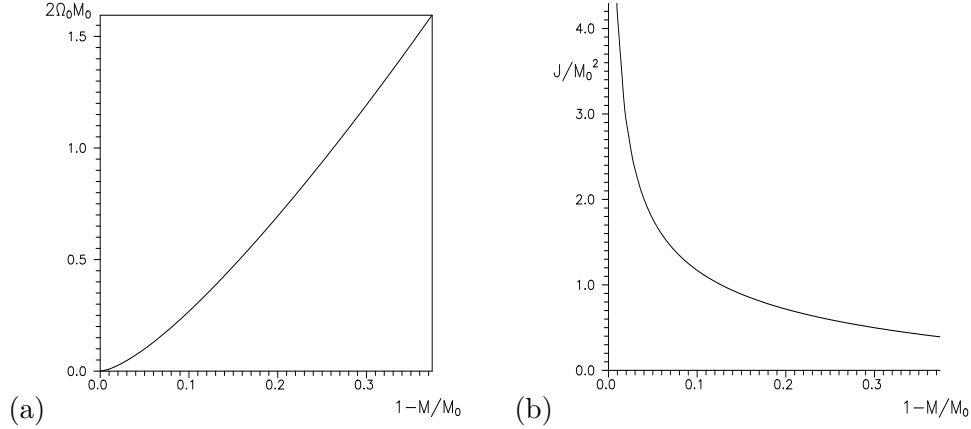


FIG. 5: (a) $\Omega_0 M_0$ and (b) J/M_0^2 as functions of the relative binding energy $(M_0 - M)/M_0$ for the disk of dust.²

relation is the angular velocity Ω_0 as a function of M and J , since Ω_0 is defined in both cases. For black holes, we have derived the explicit expression (53). Surprisingly, the corresponding disk of dust relation has the same scaling behaviour, i.e., $M\Omega_0$ is a function of M^2/J alone. Fig. 4 shows this dependence for both solutions¹⁷. For $M^2/J \rightarrow 1$ (corresponding to $\mu \rightarrow \mu_0$), where the disk solution becomes identical with the extreme Kerr solution outside the horizon ($\rho^2 + \zeta^2 > 0$), there is a “phase transition” between the disk and the black hole. Note that for non-vanishing Ω_0 , $\rho_0 \rightarrow 0$ as $\mu \rightarrow \mu_0$. A detailed analysis of the disk solution for $\mu \rightarrow \mu_0$, including the discussion of a different, non-asymptotically flat limit of space-time, which is obtained for finite ρ/ρ_0 and ζ/ρ_0 ($\rho^2 + \zeta^2 = 0$!), can be found in²⁵.

We remark that (110) solves the Bardeen-Wagoner problem²⁶ explicitly. All metric coefficients in (1) are analytic in ρ, ζ outside the disk and continuous through the disk. From a physical point of view we have an extremely flattened rigidly rotating body and, likewise, a rotating continuous distribution of mass points interacting via gravitational forces alone (‘galaxy’ model). Fig. 5 illustrates the ‘parametric’ collapse of a disk with the total mass-energy M , the baryonic mass M_0 , the angular velocity Ω_0 and the angular momentum J towards the black hole limit ($1 - M/M_0 = 0.3732835\dots$). Imagine a disk consisting of a fixed number of baryons (fixed M_0): Occupying states with decreasing energy M , it would shrink thereby shedding angular momentum but increasing its angular velocity. The above mentioned limit of the relative binding energy $1 - M/M_0$ corresponds to the extreme black hole limit. Additional physical effects (ergozones, dragging effects, surface mass density ...) as well as further parameter relations have been discussed in³ and²⁷.

The methods outlined in this paper could be used to construct self-gravitating disks around a central black hole.

APPENDIX

The metric functions e^{2U} , a , e^{2k} calculated from the Ernst potential (110) via (14)–(16) are given as follows:

$$e^{2U} = \frac{\vartheta(\mathbf{c})\vartheta^*(\mathbf{c})\vartheta(\mathbf{a})\vartheta^*(\mathbf{a})}{\vartheta(\mathbf{0})\vartheta^*(\mathbf{0})\vartheta(\mathbf{a}+\mathbf{c})\vartheta^*(\mathbf{a}+\mathbf{c})} e^{-(\gamma_0 u + \gamma_1 v + \mu w)},$$

$$1 + \frac{(1 + \Omega_0 a)e^{2U}}{\Omega_0 \rho} = \frac{\vartheta(\mathbf{0})\vartheta^*(\mathbf{0})\vartheta(\mathbf{a}+2\mathbf{c})\vartheta^*(\mathbf{a})}{\vartheta(\mathbf{c})\vartheta^*(\mathbf{0})\vartheta(\mathbf{a}+\mathbf{c})\vartheta^*(\mathbf{a}+\mathbf{c})},$$

$$e^{2k(\rho, \zeta)} = \frac{\kappa(\rho, \zeta)}{\kappa(0, 0)}$$

with

$$\kappa(\rho, \zeta) = \frac{\vartheta(\mathbf{a})\vartheta^*(\mathbf{a})}{\vartheta(\mathbf{0})\vartheta^*(\mathbf{0})} \exp \left(2k_0 - \frac{1}{2} \sum_{i,k=1}^2 a_i a_k \frac{\partial^2 \ln \vartheta(\mathbf{x})\vartheta^*(\mathbf{x})}{\partial x_i \partial x_k} \Big|_{\mathbf{x}=\mathbf{0}} \right),$$

where

$$2k_0 = \frac{\mu^2}{4} \int_{-i}^i \int_{-i}^i dX dX' \frac{(\lambda - \lambda')^2}{\lambda \lambda'} \frac{h(X)h(X')(X - X_1)(X - X_2)(X' + X_1)(X' + X_2)}{(X - X')^2},$$

$$\lambda = \sqrt{\frac{X - i\bar{z}/\rho_0}{X + iz/\rho_0}}, \quad \lambda' = \sqrt{\frac{X' - i\bar{z}/\rho_0}{X' + iz/\rho_0}},$$

$$\vartheta(\mathbf{x}) = \vartheta(\mathbf{x}; p, q, \alpha) = \vartheta(x_1, x_2; p, q, \alpha),$$

$$\vartheta^*(\mathbf{x}) = \vartheta(x_1 + \frac{i\pi}{2}, x_2 + \frac{i\pi}{2}; p, q, \alpha),$$

$$\mathbf{a} = (a_1, a_2) = (\alpha_0 u + \alpha_1 v, \beta_0 u + \beta_1 v), \quad \mathbf{0} = (0, 0), \quad \mathbf{c} = (C_1, C_2).$$

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